

Reminiscence Of An Open Problem: Remarks On Nevanlinna's Four-Value-Theorem

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Abstract. The aim of this paper is to describe the origin, first solutions, further progress, the state of art, and a new ansatz in the treatment of a problem dating back to the 1920's, which still has not found a satisfactory solution and deserves to be better known.

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1. Introduction

In [15] G. Pólya considered the problem *to determine all pairs (f, g) of distinct entire functions of finite order, such that f and g assume each of the values a_ν ($1 \leq \nu \leq 3$) at the same points and with the same multiplicities*, and solved it as follows.

Theorem (Pólya [15]). *The functions f and g have a common Picard value $a_2 = \frac{1}{2}(a_1 + a_3)$, and satisfy $(f - a_2)(g - a_2) = (a_2 - a_1)(a_3 - a_2)$.*

A typical example is $f(z) = e^z$, $g(z) = e^{-z}$, $a_1 = -1$, $a_2 = 0$, and $a_3 = 1$.

In particular, it is not possible that f and g assume each of *four* finite values at the same points and with the same multiplicities. In modern terminology Pólya's theorem says that distinct non-constant entire functions of finite order cannot *share* four finite values by counting multiplicities, and may share three finite values only in some very particular case.

The background for this theorem was as follows: if two polynomials, P and Q , say, assume integer values at the same points, then the entire functions $f = e^{2\pi i P}$ and $g = e^{2\pi i Q}$ have the Picard value 0 and share the value 1. Thus the problem arises to determine all entire functions f and g of finite order having Picard value 0 and sharing the value 1 by counting multiplicities.

Theorem (Pólya [(¹)]) *Under these hypotheses either $f = g$ or else $f = 1/g$ holds, hence either $P - Q$ or else $P + Q$ is a constant (an integer multiple of $2\pi i$).*

Since the Picard value ∞ is trivially shared by counting multiplicities, Pólya's first theorem may be looked at as a predecessor of what is nowadays known as Four-Value-Theorem due to R. Nevanlinna, while the proof of his second theorem inspired Nevanlinna to apply his theory of meromorphic functions to so-called Borel

¹We follow Nevanlinna's paper [13]; the reference "G. Pólya, Deutsche Math.-Ver. Bd. 32, S. 16, 1923" given there is incorrect.

identities. To describe these results in more detail we need some notation; familiarity with the standard notions and results of Nevanlinna's theory of meromorphic functions is assumed, see the standard references Nevanlinna [14] and Hayman [8].

Given any pair of distinct meromorphic functions f and g sharing the values \mathbf{a}_ν ($1 \leq \nu \leq q$), we set

$$T(r) = \max (T(r, f), T(r, g)),$$

and denote by $S(r)$ the usual remainder term satisfying

$$S(r) = O(\log(r T(r))) \quad (r \rightarrow \infty)$$

outside some set $E \subset (0, \infty)$ of finite measure. If f and g have finite [lower] order we have $S(r) = O(\log r)$ and E is empty [$S(r_k) = O(\log r_k)$ on some sequence $r_k \rightarrow \infty$]. Furthermore,

$$\overline{N}(r, \mathbf{a}_\nu) = \bar{n}(0, \mathbf{a}_\nu) + \log r \int_1^r [\bar{n}(t, \mathbf{a}_\nu) - \bar{n}(0, \mathbf{a}_\nu)] \frac{dt}{t}$$

denotes the (integrated) Nevanlinna counting function of the sequence of \mathbf{a}_ν -points of f and g , each point being counted simply despite of multiplicities. Then Nevanlinna's Theorems may be stated as follows:

Five-Value-Theorem (Nevanlinna [13]). *Let f and g be distinct meromorphic functions sharing the values⁽²⁾ \mathbf{a}_ν ($1 \leq \nu \leq q$). Then $q \leq 4$, and in case $q = 4$ the following is true:*

$$(N_a) \quad T(r, f) = T(r) + S(r) \text{ and } T(r, g) = T(r) + S(r).$$

$$(N_b) \quad \sum_{\nu=1}^4 \overline{N}(r, \mathbf{a}_\nu) = 2T(r) + S(r).$$

$$(N_c) \quad \overline{N}\left(r, \frac{1}{f-g}\right) = 2T(r) + S(r).^{(3)}$$

$$(N_d) \quad \overline{N}\left(r, \frac{1}{f-b}\right) = T(r) + S(r) \text{ and } \overline{N}\left(r, \frac{1}{g-b}\right) = T(r) + S(r) \quad (b \neq \mathbf{a}_\nu).$$

The typical example is the same as for Pólya's Theorem.

Four-Value-Theorem (Nevanlinna [13]). *Let f and g be non-constant meromorphic functions sharing values \mathbf{a}_ν ($1 \leq \nu \leq 4$), but now specifically by counting multiplicities. Then (relabelling the values, if necessary)*

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = (f, g, \mathbf{a}_3, \mathbf{a}_4) = -1$$

holds.⁽⁴⁾ In particular, \mathbf{a}_1 and \mathbf{a}_2 are Picard values of f and g , and g is a Möbius transformation of f that fixes \mathbf{a}_3 and \mathbf{a}_4 and permutes \mathbf{a}_1 and \mathbf{a}_2 .

Both theorems were proved by R. Nevanlinna in 1926. The novelty and importance of this paper stems from the new and powerful methods, nowadays called Nevanlinna Theory, rather than the fact that meromorphic functions of arbitrary order were considered in contrast to entire functions of finite order.

²Notably without hypothesis about the multiplicities.

³To be modified if $\mathbf{a}_4 = \infty$: $\overline{N}\left(r, \frac{1}{f-g}\right) + \overline{N}(r, \infty) = 2T(r) + S(r)$.

⁴ $(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ denotes the cross-ratio of the values a, b, c, d .

2. Progress and Counterexamples

In [4] G. Gundersen attributed the question, whether the Four-Value-Theorem also holds *without* the condition *by counting multiplicities*, to L. Rubel. This question or problem, however, was already aware to Nevanlinna, who wrote in [13]: *Es wäre nun interessant zu wissen, ob dieses Ergebnis auch dann besteht, wenn die Multiplizitäten der betreffenden Stellen nicht berücksichtigt werden. Einige im ersten Paragraphen gewonnene Ergebnisse* [here he refers to conditions (N_a) – (N_d)] *sprechen vielleicht für die Vermutung ...*⁽⁵⁾

G. Gundersen was the first to contribute to that problem. He proved the

(3+1)–Theorem (Gundersen [4]). *The conclusion of the Four-Value-Theorem remains true if f and g share four values, at least three of them by counting multiplicities.*

In the same paper, however, Gundersen also provided the first counterexample to Nevanlinna's conjecture and thus destroyed the hope for a (0+4)-Theorem. It is easily seen (and this is Gundersen's example) that *the functions*

$$(1) \quad f(z) = \frac{e^z + 1}{(e^z - 1)^2} \text{ and } g(z) = \frac{(e^z + 1)^2}{8(e^z - 1)}$$

share the values $1, 0, \infty$ and $-1/8$ in the following manner: f has only simple zeros and 1-points, and only double poles and $(-1/8)$ -points, while g has only simple poles and $(-1/8)$ -points, but has only double zeros and 1-points.

Four years later, Gundersen was again able to relax the hypothesis on the number of values shared by counting multiplicities by proving the

(2+2)–Theorem (Gundersen [5, 6]). *The conclusion of the Four-Value-Theorem remains true if f and g share four values, at least two of them by counting multiplicities.*

Gundersen's proof contained a gap, which, however, could be bridged over (see [6]) by considering the auxiliary function which E. Mues discovered in the early 1980's, but did not make use of until 1987. This function

$$\Psi = \frac{f'g'(f-g)^2}{\prod_{\nu=1}^4 (f-a_\nu)(g-a_\nu)}$$

(for $a_4 = \infty$ the factor for $\nu = 4$ has to be omitted) contains the complete information: if f and g share the values a_ν ($1 \leq \nu \leq 4$), then Ψ is an *entire* function, which is *small* in the sense that

$$T(r, \Psi) = S(r)$$

holds. It is almost trivial to deduce the 5-Theorem from this condition of smallness, and it will soon be seen that the whole progress made afterwards depends almost completely on Mues' function Ψ .

⁵*It would be interesting to know whether this result remains true regardless multiplicities. Some of the results derived in the first section seem to support the conjecture ...*

Gundersen's example may also be characterised by additional properties. This was done by M. Reinders (a student of Mues) in several directions, based on the following observations for the functions (1):

- (a) $f(z_0) = -\frac{1}{2} \Leftrightarrow g(z_0) = \frac{1}{4}$.
- (b) For ν fixed, one of the functions f and g has only \mathbf{a}_ν -points of order 2.

Theorem (Reinders [16, 17]). *Assume that f and g share mutually distinct values \mathbf{a}_ν ($1 \leq \nu \leq 4$), and that one of the following conditions holds:*

- (a) *There exist values $a, b \neq \mathbf{a}_\nu$ ($1 \leq \nu \leq 4$) such that $f(z_0) = a$ implies $g(z_0) = b$, and the conclusion of the Four-Value-Theorem does not hold.*
- (b₁) *For every ν the zeros of $(f - \mathbf{a}_\nu)(g - \mathbf{a}_\nu)$ have multiplicity 3.*
- (b₂) *For every ν either $f - \mathbf{a}_\nu$ or else $g - \mathbf{a}_\nu$ has only zeros of order 2.*

Then up to pre-composition with some non-constant entire function h , and post-composition with some Möbius transformation, f and g coincide with the functions in Gundersen's example (1).

We note that conditions (b₁) and (b₂) look quite different (although they turn out *a posteriori* to be equal), since in case (b₁), even for fixed ν , each of f and g may have double \mathbf{a}_ν -points, while this is not the case in (b₂).

All attempts to prove or disprove a (1+3)-Theorem failed up to now (January 2011). Before describing further progress we mention a counterexample that is quite different from Gundersen's and has a different origin. It arose from a Comptes Rendus note of H. Cartan [2], where, in modern terminology, the following was stated: *There do not exist three mutually distinct meromorphic functions sharing four values.* The proof indicated in [2], however, contained a serious gap, as Mues pointed out to the author in the early 1980's. This time the gap could not be bridged over, and it took more than three years to find a way leading to the true statement and, by the way, to a new counterexample characterised by that theorem.

Triple Theorem (Steinmetz [21]). *Suppose that three mutually distinct meromorphic functions f, g, h share four values, $0, 1, \infty$ and \mathbf{a} , say. Then*

- $\mathbf{a} \neq -1$ is a third root of -1 , and
- $w = f(z), g(z), h(z)$ are solutions to the algebraic equation

$$(2) \quad w^3 + 3[(\bar{\mathbf{a}} + 1)u^2(z) + 2u(z)]w^2 - 3[2u^2(z) + (\mathbf{a} + 1)u(z)]w - u^3(z) = 0.$$

Here $u = v \circ \gamma$, where γ is some non-constant entire function and v some non-constant solution to the differential equation

$$(3) \quad (v')^2 = 4v(v + 1)(v - \mathbf{a}).$$

Conversely, given $\mathbf{a} \neq 0, -1$, the solutions to equation (3) are elliptic functions, and for $\mathbf{a} = \frac{1}{2}(1 \pm i\sqrt{3})$ and any non-constant entire function γ , the solutions to equation (2) with $u = v \circ \gamma$ provides three meromorphic functions sharing the values $0, 1, \infty$, and \mathbf{a} .

In the most simple case $\gamma(z) = z$ these functions are elliptic functions of elliptic order six. They share the values $0, 1, \infty$, and \mathbf{a} in the following manner: every

period parallelogram contains three c -points ($c \in \{0, 1, \infty, \mathbf{a}\}$), each being simple for two of these functions, and having multiplicity *four* for the third one. Thus the sequence of c -points is divided in a natural way into three subsequences having asymptotically equal counting functions.

Reinders constructed a counterexample that is quite different from Gundersen's. It is well-known that the non-constant solutions of the differential equation

$$(4) \quad (u')^2 = 12u(u+1)(u+4),$$

are elliptic functions of elliptic order two (actually $u(z) = \wp(\sqrt{3}z + c) - 5/3$, where \wp is the specific P-function of Weierstrass that satisfies the differential equation $(\wp')^2 = 4(\wp - \frac{5}{3})(\wp - \frac{2}{3})(\wp + \frac{7}{3})$). *Then for any such function,*

$$(5) \quad f = \frac{1}{8\sqrt{3}} \frac{uu'}{u+1} \text{ and } g = \frac{1}{8\sqrt{3}} \frac{(u+4)u'}{(u+1)^2}$$

share the values $-1, 0, 1, \infty$ in the following manner: each of these values is assumed in an alternating way with multiplicity 1 by one of the functions f and g , and with multiplicity 3 by the other one (e.g., f has triply zeros at the zeros of u , and simple zeros when $u+4=0$.)

Again this example can be characterised by this particular property.

Theorem (Reinders [18]). *Let f and g be meromorphic functions sharing finite values \mathbf{a}_ν ($1 \leq \nu \leq 4$), such that $(f - \mathbf{a}_\nu)(g - \mathbf{a}_\nu)$ has always zeros of multiplicity at least 4. Then up to pre-composition with some non-constant entire function and post-composition with some Möbius transformation, the functions f and g coincide with those in example (5).*

Remark. Suppose that there exist positive integers $p < q$, such that every shared value \mathbf{a}_ν is assumed by f with multiplicity p and by g with multiplicity q , or vice versa. Then considering Mues' function yields $p = 1$, and from

$$(1+q)\overline{N}(r, \mathbf{a}_\nu) = N\left(r, \frac{1}{f - \mathbf{a}_\nu}\right) + N\left(r, \frac{1}{g - \mathbf{a}_\nu}\right) + S(r) \leq 2T(r) + S(r)$$

and condition (N_b) easily follows $2(1+q)T(r) \leq 8T(r) + S(r)$, hence $2 \leq q \leq 3$. On combination with the theorems of Reinders this immediately yields:

Theorem (Chen, Chen, Ou & Tsai [3]). *Suppose f and g share four values \mathbf{a}_ν , and assume also that there are integers $1 \leq p < q$ such that each \mathbf{a}_ν -point is either a (p, q) -fold point for (f, g) or for (g, f) . Then the conclusion of Reinders' theorems [16, 17, 18] hold.*

3. Proof of the (2+2)–Theorem

Any progress till now relies on the fact that “(2+2) implies (4+0)”, and was based on Mues' auxiliary function technique [10]. To show the power of this method we will next give independent and short proofs of the Four-Value-, the (3+1)-, and the (2+2)–Theorem.

Proof of the (4+0)–Theorem—Mues [10]. Suppose f and g share four values \mathbf{a}_ν ($1 \leq \nu \leq 4$) by counting multiplicities. Then at least two of the values \mathbf{a}_ν satisfy $\overline{N}(r, \mathbf{a}_\nu) \neq S(r)$. We may assume that $\mathbf{a}_4 = \infty$ does, hence $N(r, f) = \overline{N}(r, \infty) + S(r)$ and $N(r, g) = \overline{N}(r, \infty) + S(r)$ hold, while $N(r, 1/f') + N(r, 1/g') = S(r)$ is an easy consequence of the strong assumption “by counting multiplicities”. Thus the auxiliary function

$$\phi = \frac{f''}{f'} - \frac{g''}{g'}$$

satisfies $T(r, \phi) = S(r)$, but vanishes at all poles that are simple for f and g . This yields a contradiction, namely $\overline{N}(r, \infty) = S(r)$, if $\phi \not\equiv 0$. If, however, ϕ vanishes identically, $f = g$ follows at once. \square

Proof of the (3+1)–Theorem—Rudolph [19]. Suppose f and g share the finite values \mathbf{a}_ν ($1 \leq \nu \leq 3$) by counting multiplicities, and $\mathbf{a}_4 = \infty$ without further hypothesis. Then

$$\phi = \frac{f'}{\prod_{\nu=1}^3 (f - \mathbf{a}_\nu)} - \frac{g'}{\prod_{\nu=1}^3 (g - \mathbf{a}_\nu)}$$

satisfies $T(r, \phi) = S(r)$ and vanishes at the poles of f and g . Thus we have either $\phi \not\equiv 0$ and $\overline{N}(r, \infty) = S(r)$, or else ϕ vanishes identically, the latter meaning that also $\mathbf{a}_4 = \infty$ is shared by counting multiplicities. On the other hand it is not hard to show that the hypotheses “ $\overline{N}(r, \infty) = S(r)$ ” and “ $\mathbf{a}_4 = \infty$ is shared by counting multiplicities” are equally strong in the sense that (on combination with the hypotheses about the values \mathbf{a}_ν) they lead to the same conclusions. \square

Proof of the (2+2)–Theorem. We may assume that the values 0 and ∞ are shared by counting multiplicities, while the other values are \mathbf{a}_1 and $\mathbf{a}_2 = 1/\mathbf{a}_1$ (if the latter does not hold *a priori*, we consider cf and cg instead of f and g , with c satisfying $c^2 \mathbf{a}_1 \mathbf{a}_2 = 1$). The auxiliary function

$$\phi = \frac{f''}{f'} + 2\frac{f'}{f} - \sum_{\nu=1}^2 \frac{f'}{f - \mathbf{a}_\nu} - \frac{g''}{g'} - 2\frac{g'}{g} + \sum_{\nu=1}^2 \frac{g'}{g - \mathbf{a}_\nu}$$

is regular at all \mathbf{a}_ν -points ($\nu = 1, 2$) of f and g despite their multiplicities, and is also regular at the zeros and poles that are simple for f and g . Thus ϕ satisfies $T(r, \phi) = S(r)$, and

$$\phi(z_\infty)^2 = (\mathbf{a}_1 + \mathbf{a}_2)^2 \Psi(z_\infty)$$

holds at any pole which is simple for f and g ; here Ψ again is Mues' function. Hence either $\overline{N}(r, \infty) = S(r)$ or else $\phi^2 = (\mathbf{a}_1 + \mathbf{a}_2)^2 \Psi$ holds. Repeating this argument with $F = 1/f$, $G = 1/g$ and corresponding function

$$\begin{aligned} \chi &= \frac{F''}{F'} + 2\frac{F'}{F} - \sum_{\nu=1}^2 \frac{F'}{F - 1/\mathbf{a}_\nu} - \frac{G''}{G'} - 2\frac{G'}{G} + \sum_{\nu=1}^2 \frac{G'}{G - 1/\mathbf{a}_\nu} \\ &= \frac{f''}{f'} - 2\frac{f'}{f} - \sum_{\nu=1}^2 \frac{f'}{f - \mathbf{a}_\nu} - \frac{g''}{g'} + 2\frac{g'}{g} + \sum_{\nu=1}^2 \frac{g'}{g - \mathbf{a}_\nu} \end{aligned}$$

instead of f , g and ϕ , and noting that $1/\mathbf{a}_1 + 1/\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_2$ and Ψ is also Mues' function for F and G , it follows that essentially four possibilities remain to be discussed:

- (a) $\overline{N}(r, 0) + \overline{N}(r, \infty) = S(r)$;
- (b) $\phi = \chi$;
- (c) $\phi = -\chi$;
- (d) $\overline{N}(r, \infty) = S(r)$ and $\chi^2 = (\mathbf{a}_1 + \mathbf{a}_2)^2 \Psi$.

ad (a)—From (N_b) follows $\overline{N}(r, \mathbf{a}_1) + \overline{N}(r, \mathbf{a}_2) = 2T(r) + S(r)$, hence the sequence of \mathbf{a}_ν -points ($\nu = 1, 2$) which have different multiplicities for f and g have counting function $S(r)$. The conclusion of the Four-Value-Theorem follows immediately from Mues' proof of that theorem.

ad (b)—From $\phi - \chi = 4\left(\frac{f'}{f} - \frac{g'}{g}\right) = 0$ follows $g = cf$, hence $g = f$ if there exists some common \mathbf{a}_ν -point. Otherwise \mathbf{a}_1 and \mathbf{a}_2 are Picard values for f and g , hence the Four-Value-Theorem holds.

ad (c)—Here we have $\frac{f''}{f'} - \sum_{\nu=1}^2 \frac{f'}{f - \mathbf{a}_\nu} - \frac{g''}{g'} + \sum_{\nu=1}^2 \frac{g'}{g - \mathbf{a}_\nu} = 0$, hence

$$\frac{f'}{(f - \mathbf{a}_1)(f - \mathbf{a}_2)} = \kappa \frac{g'}{(g - \mathbf{a}_1)(g - \mathbf{a}_2)} \quad (\kappa \neq 0)$$

and

$$\frac{f - \mathbf{a}_1}{f - \mathbf{a}_2} = C \left(\frac{g - \mathbf{a}_1}{g - \mathbf{a}_2} \right)^\kappa$$

for some $\kappa \in \mathbb{Z} \setminus \{0\}$ and $C \neq 0$. Then $\kappa = \pm 1$ follows from $T(r, f) = T(r, g) + S(r)$, and two cases remain to be discussed: $\kappa =$

- 1. The values \mathbf{a}_ν are Picard values for f and g , hence f and g share all values by counting multiplicities.
- 1. From $\frac{f - \mathbf{a}_1}{f - \mathbf{a}_2} = C \frac{g - \mathbf{a}_1}{g - \mathbf{a}_2}$ follows that f and g also share the values \mathbf{a}_1 and \mathbf{a}_2 , hence all values, by counting multiplicities.

ad (d)—Following Mues [10] we consider the auxiliary functions

$$\eta_1 = \chi - (\mathbf{a}_1 + \mathbf{a}_2) \frac{f'(f - g)}{f(g - \mathbf{a}_1)(f - \mathbf{a}_2)} \text{ and } \theta_1 = \chi - (\mathbf{a}_1 + \mathbf{a}_2) \frac{g'(f - g)}{g(f - \mathbf{a}_1)(g - \mathbf{a}_2)}$$

and the corresponding functions η_2 and θ_2 (with \mathbf{a}_1 and \mathbf{a}_2 permuted). It is easily seen that

$$T(r, \eta_\nu) \leq T(r) - \overline{N}(r, \mathbf{a}_\nu) + S(r) \text{ and } T(r, \theta_\nu) \leq T(r) - \overline{N}(r, \mathbf{a}_\nu) + S(r)$$

holds (see also [10]). Now each of these functions vanishes at the zeros of f and g . If the functions η_1 and θ_1 as well as η_2 and θ_2 do not simultaneously vanish identically, we obtain (note that $\overline{N}(r, \infty) = S(r)$)

$$2\overline{N}(r, 0) \leq 2T(r) - \overline{N}(r, \mathbf{a}_1) - \overline{N}(r, \mathbf{a}_2) + S(r) = \overline{N}(r, 0) + S(r),$$

hence $\overline{N}(r, 0) = S(r)$. So we are back in case (a), and the (2+2)-Theorem is proved completely in that case.

On the other hand, $\eta_1 = \theta_1 \equiv 0$, say, and $\mathbf{a}_1 + \mathbf{a}_2 \neq 0$ yield

$$\frac{(f - \mathbf{a}_1)f'}{f(f - \mathbf{a}_2)} = \frac{(g - \mathbf{a}_1)g'}{g(g - \mathbf{a}_2)},$$

hence f and g share the value \mathbf{a}_1 by counting multiplicities, so that the (3+1)–Theorem gives the desired result.

The case $\mathbf{a}_1 + \mathbf{a}_2 = 0$ (hence $\mathbf{a}_1 = i$ and $\mathbf{a}_2 = -i$, on combination with $\mathbf{a}_1 \mathbf{a}_2 = 1$) has to be treated separately. From $\chi \equiv 0$ follows

$$(6) \quad \frac{f'}{f^2(f-i)(f+i)} = \kappa \frac{g'}{g^2(g-i)(g+i)} \quad (\kappa \neq 0).$$

Since the values $\pm i$ are not Picard values for f and g (otherwise we were already in the (3+1)-case), κ (or $1/\kappa$) is a positive integer, hence f assumes the values $\pm i$ “always” with multiplicity κ , while g has “only” simple $\pm i$ -points (up to a sequence of points with counting function $S(r)$). Now (6) is equivalent to

$$\frac{f'}{f^2} - \kappa \frac{g'}{g^2} = \frac{i}{2} \left(\frac{f'}{f-i} - \frac{f'}{f+i} - \kappa \frac{g'}{g-i} + \kappa \frac{g'}{g+i} \right),$$

and the right hand side, denoted φ , is regular at $\pm i$ -points and vanishes at simple zeros z_0 of f and g (note that $g'(z_0) = \kappa f'(z_0)$). Also from $T(r, \varphi) = S(r)$ follows either $\overline{N}(r, 0) = S(r)$, which leads back to case (a), or else $\varphi \equiv 0$, equivalently

$$\frac{f'}{f^2} - \kappa \frac{g'}{g^2} = 0 \text{ and } \frac{1}{f} = \frac{\kappa}{g} + c$$

with $\pm 1/i = \pm \kappa/i + c$. This is only possible if $\kappa = 1$ and $c = 0$. \square

4. Progress after Gundersen

As already mentioned, all attempts to prove or disprove a (1+3)–Theorem failed up to now (January 2011), and so many authors looked for additional conditions or switched to related problems—but the latter is not the subject of the present paper. In [10] E. Mues introduced the quantity

$$\tau(\mathbf{a}_\nu) = \liminf_{r \rightarrow \infty \atop (r \notin E)} \frac{N_s(r, \mathbf{a}_\nu)}{\overline{N}(r, \mathbf{a}_\nu)} \quad \text{if } \overline{N}(r, \mathbf{a}_\nu) \neq S(r),$$

and $\tau(\mathbf{a}_\nu) = 1$ else; here $N_s(r, \mathbf{a}_\nu)$ denotes the counting function of those \mathbf{a}_ν -points which are *simultaneously simple* for f and g , and E is the exceptional set for $S(r)$; we note that $\tau(\mathbf{a}_\nu) = 1$ in particular holds if \mathbf{a}_ν is shared by counting multiplicities, and also if \mathbf{a}_ν is a Picard value for f and g .

Example. We have $\tau(\mathbf{a}_\nu) = 0$ in the counterexamples of Gundersen and Reinders, and $\tau(\mathbf{a}_\nu) = \frac{1}{3}$ for any pair of the author’s triple.

In her diploma thesis, E. Rudolph [19] proved some results in terms of the quantities $\tau(\mathbf{a}_\nu)$ or their natural generalisations $\tau(\mathbf{a}_\nu, \mathbf{a}_\mu)$, $\tau(\mathbf{a}_\nu, \mathbf{a}_\mu, \mathbf{a}_\kappa)$ and $\tau(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$. Her proofs were based on the methods developed in the paper [10], which appeared later than [19], but was written earlier. In the sequel we shall derive several results, which become much more apparent when stated as inequalities involving the counting functions $\overline{N}(r, \mathbf{a}_\nu)$ and $N_s(r, \mathbf{a}_\nu)$. Nevertheless they may be credited to Rudolph and Mues for the underlying idea.

Key Lemma. *Suppose that f and g share distinct values \mathbf{a}_ν ($1 \leq \nu \leq 4$). Then either the conclusion of the Four-Value-Theorem holds or else the following is true:*

$$\begin{aligned}
(R_a) \quad & \frac{3}{2}N_{\mathfrak{s}}(r, \mathbf{a}_{\kappa}) + N_{\mathfrak{s}}(r, \mathbf{a}_{\nu}) \leq \overline{N}(r, \mathbf{a}_{\kappa}) + \overline{N}(r, \mathbf{a}_{\nu}) + S(r) \text{ for } \kappa \neq \nu; \\
& \text{for } (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1 \text{ the factor } \frac{3}{2} \text{ can be replaced by } 2. \\
(R_b) \quad & N_{\mathfrak{s}}(r, \mathbf{a}_{\kappa}) + \sum_{\mu \neq \lambda} N_{\mathfrak{s}}(r, \mathbf{a}_{\mu}) \leq \sum_{\mu \neq \lambda} \overline{N}(r, \mathbf{a}_{\mu}) + S(r) \text{ for } \kappa \neq \lambda. \\
(R_c) \quad & \overline{N}(r, \mathbf{a}_{\kappa}) + \sum_{\mu \neq \kappa} N_{\mathfrak{s}}(r, \mathbf{a}_{\mu}) \leq \sum_{\mu \neq \kappa} \overline{N}(r, \mathbf{a}_{\mu}) + S(r). \\
(R_d) \quad & \sum_{\mu=1}^4 N_{\mathfrak{s}}(r, \mathbf{a}_{\mu}) + \sum_{\mu \neq \kappa, \nu} N_{\mathfrak{s}}(r, \mathbf{a}_{\mu}) \leq 2 \sum_{\mu \neq \kappa, \nu} \overline{N}(r, \mathbf{a}_{\mu}) \text{ for } \kappa \neq \nu. \\
(R_e) \quad & 4 \sum_{\kappa \neq \lambda} N_{\mathfrak{s}}(r, \mathbf{a}_{\kappa}) \leq 3 \sum_{\kappa \neq \lambda} \overline{N}(r, \mathbf{a}_{\kappa}) + S(r). \\
(R_f) \quad & 3 \sum_{\mu=1}^4 N_{\mathfrak{s}}(r, \mathbf{a}_{\mu}) \leq 2 \sum_{\mu=1}^4 \overline{N}(r, \mathbf{a}_{\mu}) + S(r) = 4T(r) + S(r).
\end{aligned}$$

Several more or less recent results obtained by different authors can easily be derived from the previous lemma. Since, however, the thesis [19] has never been published, these results can be looked upon as independent discoveries. The proof of the Key Lemma will be given in the next section.

Corollary. *Suppose that f and g share the values \mathbf{a}_{ν} ($1 \leq \nu \leq 4$). Then the conclusion of the Four-Value-Theorem is true, provided one of the following hypotheses is assumed in addition:*

- (A) One value is shared by counting multiplicities, and some other satisfies $\tau(\mathbf{a}_{\kappa}) > \frac{2}{3}$; for $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1$ the constant $\frac{2}{3}$ can be replaced by $\frac{1}{2}$.—(E. Mues [10]).⁽⁶⁾
- (B) One value is shared by counting multiplicities, while the other values satisfy $\tau(\mathbf{a}_{\nu}) > \frac{1}{2}$.—(J.P. Wang [24], B. Huang [9]).
- (C) One value is shared by counting multiplicities and simultaneously satisfies $\overline{N}(r, \mathbf{a}_{\kappa}) \geq (\frac{4}{5} + \delta)T(r)$ for some $\delta > 0$ on some set of infinite measure.—(G. Gundersen [7]).
- (D) Two of the values satisfy $\tau(\mathbf{a}_{\nu}) > \frac{4}{5}$; for $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1$ the constant $\frac{4}{5}$ can be replaced by $\frac{2}{3}$.—(E. Rudolph [19], S.P. Wang [23], B. Huang [9]).
- (E) Three of the values satisfy $\tau(\mathbf{a}_{\nu}) > \frac{3}{4}$.—(E. Rudolph [19], G. Song & J. Chang [20]).
- (F) All values satisfy $\tau(\mathbf{a}_{\nu}) > \frac{2}{3}$.—(E. Rudolph [19], H. Ueda [22])⁽⁷⁾, J.P. Wang [24], also conjectured by G. Song & J. Chang [20]).

Proof. Assuming that the conclusion of Nevanlinna's Four-Value-Theorem does *not* hold, we will derive a contradiction to the respective hypothesis by applying one of the inequalities (R_a) – (R_f).

⁶It is remarkable that Mues did not refer to Gundersen's (2+2)-Theorem, hence, in particular, gave an independent proof for it. The other authors actually proved that the hypotheses of the (2+2)-Theorem follow from their own, thus their theorems generalising Gundersen's (2+2)-result actually depend on it.

⁷Ueda has a slightly stronger result involving some additional term, which, however, cannot be controlled.

(A) is true since

$$\overline{N}(r, \mathbf{a}_\nu) = N_{\mathfrak{s}}(r, \mathbf{a}_\nu) + S(r)$$

for some ν and (R_a) for the same ν imply

$$\frac{3}{2}N_{\mathfrak{s}}(r, \mathbf{a}_\kappa) \leq \overline{N}(r, \mathbf{a}_\kappa) + S(r) \quad (\kappa \neq \nu),$$

with $\frac{3}{2}$ replaced by 2 if $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1$.

To prove (B) we may assume that \mathbf{a}_4 is shared by counting multiplicities, thus

$$\overline{N}(r, \mathbf{a}_4) = N_{\mathfrak{s}}(r, \mathbf{a}_4) + S(r)$$

holds. From (R_c) with $\kappa < 4$ then follows

$$\overline{N}(r, \mathbf{a}_\kappa) + \sum_{\mu \neq \kappa, 4} N_{\mathfrak{s}}(r, \mathbf{a}_\mu) \leq \sum_{\mu \neq \kappa, 4} \overline{N}(r, \mathbf{a}_\mu) + S(r),$$

and adding up for $\kappa = 1, 2, 3$ yields

$$\sum_{\kappa=1}^3 \overline{N}(r, \mathbf{a}_\kappa) + 2 \sum_{\mu=1}^3 N_{\mathfrak{s}}(r, \mathbf{a}_\mu) \leq 2 \sum_{\mu=1}^3 \overline{N}(r, \mathbf{a}_\mu) + S(r).$$

(C) is obtained from (R_a) as follows. We assume that \mathbf{a}_κ is shared by counting multiplicities, hence $\overline{N}(r, \mathbf{a}_\kappa) = N_{\mathfrak{s}}(r, \mathbf{a}_\kappa) + S(r)$ holds. Adding up for $\nu \neq \kappa$ we obtain

$$\frac{3}{2}\overline{N}(r, \mathbf{a}_\kappa) + \sum_{\nu \neq \kappa} N_{\mathfrak{s}}(r, \mathbf{a}_\nu) \leq \sum_{\nu \neq \kappa} \overline{N}(r, \mathbf{a}_\nu) + S(r) = 2T(r) - \overline{N}(r, \mathbf{a}_\kappa) + S(r).$$

(D) again follows from (R_a) by adding up the symmetric inequalities

$$\begin{aligned} \frac{3}{2}N_{\mathfrak{s}}(r, \mathbf{a}_\nu) + N_{\mathfrak{s}}(r, \mathbf{a}_\kappa) &\leq \overline{N}(r, \mathbf{a}_\kappa) + \overline{N}(r, \mathbf{a}_\nu) + S(r) \\ \frac{3}{2}N_{\mathfrak{s}}(r, \mathbf{a}_\kappa) + N_{\mathfrak{s}}(r, \mathbf{a}_\nu) &\leq \overline{N}(r, \mathbf{a}_\kappa) + \overline{N}(r, \mathbf{a}_\nu) + S(r). \end{aligned}$$

Again we note that $\frac{4}{5}$ can be replaced by $\frac{2}{3}$ if $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1$.

Finally, (E) and (F) follow from (R_e) and (R_f) , respectively. \square

5. Proof of the Key Lemma

The proof of the Key Lemma is based on Mues' auxiliary function technique [10]. As long as no particular hypotheses are imposed, the proofs and results are symmetric and Möbius invariant, this meaning that everything proved for $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ holds for arbitrary permutations, and that three of the four values can be given prescribed numerical values.

To prove (R_a) we proceed as in the proof of the (2+2)–Theorem. We set $\kappa = 4$, $\mathbf{a}_4 = \infty$, and $\nu = 3$, and consider Mues' function Ψ and the auxiliary function

$$\phi = \frac{f''}{f'} + 2 \frac{f'}{f - \mathbf{a}_3} - \sum_{\nu=1}^2 \frac{f'}{f - \mathbf{a}_\nu} - \frac{g''}{g'} - 2 \frac{g'}{g - \mathbf{a}_3} + \sum_{\nu=1}^2 \frac{g'}{g - \mathbf{a}_\nu}.$$

Then ϕ has only simple poles, exactly at those \mathbf{a}_3 -points and poles (\mathbf{a}_4 -points) that are *not* simultaneously simple for f and g , and the usual technique yields

$$T(r, \phi) = \overline{N}(r, \mathbf{a}_3) - N_{\mathfrak{s}}(r, \mathbf{a}_3) + \overline{N}(r, \mathbf{a}_4) - N_{\mathfrak{s}}(r, \mathbf{a}_4) + S(r).$$

Since

$$\phi(z_\infty)^2 = (2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2)^2 \Psi(z_\infty)$$

holds at every pole z_∞ that is simple for f and g ,

$$\begin{aligned} N_s(r, \mathbf{a}_4) &\leq 2T(r, \phi) + O(1) \\ &\leq 2(\overline{N}(r, \mathbf{a}_3) - N_s(r, \mathbf{a}_3) + \overline{N}(r, \mathbf{a}_4) - N_s(r, \mathbf{a}_4)) + S(r) \end{aligned}$$

and (R_a) for $\kappa = 4$ and $\nu = 3$ follow, provided $\phi^2 \neq (2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2)^2 \Psi$ holds. On the other hand, $\phi^2 = (2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2)^2 \Psi$ implies that ϕ is a small function, thus f and g share the values \mathbf{a}_3 and $\mathbf{a}_4 = \infty$ by counting multiplicities, and hence all values \mathbf{a}_ν by the (2+2)-Theorem. For $2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2 = 0$ (equivalently $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \infty) = -1$) we obtain the better inequality

$$2N_s(r, \mathbf{a}_4) + N_s(r, \mathbf{a}_3) \leq \overline{N}(r, \mathbf{a}_3) + \overline{N}(r, \mathbf{a}_4) + S(r),$$

by counting the zeros of ϕ rather than those of $\phi^2 - (2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2)^2 \Psi$.

To prove (R_b) we may assume $\kappa = 4$, $\mathbf{a}_4 = \infty$ and $\lambda = 3$. Then

$$\phi = \frac{f''}{f'} - 2 \frac{f'}{f - \mathbf{a}_3} + \frac{f'(f - \mathbf{a}_3)}{(f - \mathbf{a}_1)(f - \mathbf{a}_2)} - \frac{g''}{g'} - 2 \frac{g'}{g - \mathbf{a}_3} - \frac{g'(g - \mathbf{a}_3)}{(g - \mathbf{a}_1)(g - \mathbf{a}_2)}$$

is regular at poles of f and g ($\mathbf{a}_4 = \infty$), has simple poles exactly at those \mathbf{a}_ν -points ($1 \leq \nu \leq 3$) of f and g that have different multiplicities, thus

$$(7) \quad T(r, \phi) = N(r, \phi) + S(r) = \sum_{\nu=1}^3 (\overline{N}(r, \mathbf{a}_\nu) - \overline{N}_s(r, \mathbf{a}_\nu)) + S(r)$$

holds. Now ϕ vanishes at \mathbf{a}_3 -points which are simultaneously simple for f and g , hence $\phi \not\equiv 0$ implies

$$N_s(r, \mathbf{a}_3) \leq \sum_{\nu=1}^3 (\overline{N}(r, \mathbf{a}_\nu) - N_s(r, \mathbf{a}_\nu)) + S(r),$$

thus (R_b) for $\kappa = 4$ and $\lambda = 3$. On the other hand, the conclusion of the Four-Value-Theorem holds if ϕ vanishes identically.

To prove (R_c) , for $\kappa = 4$ and $\mathbf{a}_4 = \infty$, say, we just consider

$$\phi = \frac{f'}{(f - \mathbf{a}_1)(f - \mathbf{a}_2)(f - \mathbf{a}_3)} - \frac{g'}{(g - \mathbf{a}_1)(g - \mathbf{a}_2)(g - \mathbf{a}_3)};$$

then ϕ has poles exactly at those \mathbf{a}_ν -points ($1 \leq \nu \leq 3$) that have different multiplicities for f and g , and thus satisfies *a fortiori* (7). Since ϕ vanishes at poles (\mathbf{a}_4 -points) of f and g , this yields

$$\overline{N}(r, \mathbf{a}_4) \leq \sum_{\nu=1}^3 (\overline{N}(r, \mathbf{a}_\nu) - N_s(r, \mathbf{a}_\nu)) + S(r)$$

provided ϕ does not vanish identically. If, however, $\phi = 0$, then f and g share the values \mathbf{a}_μ ($1 \leq \mu \leq 3$) by counting multiplicities, and thus the hypothesis of the (3+1)-Theorem and the conclusion of the Four-Value-Theorem holds.

To prove (R_d) we may assume $\mathbf{a}_\mu \in \mathbb{C}$ ($1 \leq \mu \leq 4$) and consider

$$\phi = \frac{f'}{(f - \mathbf{a}_1)(f - \mathbf{a}_2)} - \frac{g'}{(g - \mathbf{a}_1)(g - \mathbf{a}_2)}.$$

Then ϕ has poles exactly at those \mathbf{a}_1 - and \mathbf{a}_2 -points that are not simultaneously simple for f and g , and is regular at poles of f and of g , hence

$$T(r, \phi) = \sum_{\mu=1}^2 (\overline{N}(r, \mathbf{a}_\mu) - N_{\mathbf{s}}(r, \mathbf{a}_\mu)) + S(r)$$

follows. On the other hand, $\phi(z_\rho) = (-1)^\rho \frac{f'(z_\rho) - g'(z_\rho)}{\prod_{\mu \neq \rho} (\mathbf{a}_\rho - \mathbf{a}_\mu)} (\mathbf{a}_4 - \mathbf{a}_3)$ holds at any

\mathbf{a}_ρ -point z_ρ ($\rho = 3, 4$), and $\Psi(z_\rho) = \frac{(f'(z_\rho) - g'(z_\rho))^2}{\prod_{\mu \neq \rho} (\mathbf{a}_\rho - \mathbf{a}_\mu)^2}$ holds whenever z_ρ is simple

for f and g . Thus if $\phi^2 \neq (\mathbf{a}_4 - \mathbf{a}_3)^2 \Psi$, the assertion (for $\kappa = 3$, $\nu = 4$) follows from the First Main Theorem of Nevanlinna:

$$\begin{aligned} \sum_{\mu=3}^4 N_{\mathbf{s}}(r, \mathbf{a}_\mu) &\leq T\left(r \frac{1}{\phi^2 - (\mathbf{a}_4 - \mathbf{a}_3)^2 \Psi}\right) + O(1) = 2T(r, \phi) + S(r) \\ &\leq 2 \sum_{\mu=1}^2 (\overline{N}(r, \mathbf{a}_\mu) - N_{\mathbf{s}}(r, \mathbf{a}_\mu)) + S(r). \end{aligned}$$

If, however, $\phi^2 = (\mathbf{a}_4 - \mathbf{a}_3)^2 \Psi$, then f and g share the values \mathbf{a}_1 and \mathbf{a}_2 by counting multiplicities, hence the conclusion of the Four-Value-Theorem holds.

Finally, (R_e) and (R_f) follow by adding up inequality (R_b) for $\kappa \neq \lambda$ and (R_c) for $\kappa = 1, \dots, 4$, respectively. \square

6. Towards or way off a (1+3)-Theorem?

Assuming that f and g share the values $\mathbf{a}_\nu \in \mathbb{C}$ ($1 \leq \nu \leq 3$) and $\mathbf{a}_4 = \infty$, we set

$$\begin{aligned} \phi_f &= \frac{f'}{(f - \mathbf{a}_1)(f - \mathbf{a}_2)(f - \mathbf{a}_3)} & \text{and} & & \phi_g &= \frac{g'}{(g - \mathbf{a}_1)(g - \mathbf{a}_2)(g - \mathbf{a}_3)}, \\ \Phi_f &= (f - g)\phi_f & \text{and} & & \Phi_g &= (f - g)\phi_g, \\ \Phi &= \Phi_f/\Phi_g = \phi_f/\phi_g & \text{and} & & \Psi &= \Phi_f\Phi_g \text{ (Mues' function)}. \end{aligned}$$

functions f, g	Φ_f	Φ_g	Ψ	Φ
e^z, e^{-z}	e^{-z}	e^z	1	e^{-2z}
Gundersen's	$1 - e^z$	$\frac{8}{1 - e^z}$	8	$\frac{(1 - e^z)^2}{8}$
Reinders'	$\frac{12\sqrt{3}}{u+1}$	$\frac{12(u+1)}{\sqrt{3}}$	144	$\frac{9}{(u+1)^2}$

Example.

Key Observations.

- If the value $\mathbf{a}_4 = \infty$ is shared by counting multiplicities, then Φ_f and Φ_g are entire functions satisfying $N(r, 1/\Phi_f) + N(r, 1/\Phi_g) = S(r)$.
- If $T(r)$ has finite lower order $\liminf_{r \rightarrow \infty} \frac{\log T(r)}{\log r}$, then

$$\Phi_f = p_f e^Q, \quad \Phi_g = p_g e^{-Q}, \quad \Psi = p_f p_g, \quad \Phi = \frac{p_f}{p_g} e^{2Q}, \quad \text{and} \quad T(r) \asymp r^{\deg Q}$$

hold with polynomials p_f , p_g , and Q .

Proof. Noting that

$$\Psi = \frac{f^2 f'}{P(f)} \frac{g'}{P(g)} + 2 \frac{f f'}{P(f)} \frac{g g'}{P(g)} + \frac{f'}{P(f)} \frac{g^2 g'}{P(g)}$$

(with $P(w) = \prod_{\nu=1}^3 (w - \mathbf{a}_\nu)$ or $P(w) = \prod_{\nu=1}^4 (w - \mathbf{a}_\nu)$) holds, the lemma on the logarithmic derivative gives

$$T(r_k, \Psi) = m(r_k, \Psi) = O(\log r_k)$$

on some sequence $r_k \rightarrow \infty$. Hence Ψ is a polynomial, Φ_f and Φ_g have only finitely many zeros and satisfy

$$\log T(r_k, \Phi_f) + \log T(r_k, \Phi_g) = O(\log r_k).$$

Thus the assertion on Φ_f , Φ_g and Ψ , holds, since entire functions $e^{h(z)}$ have finite lower order if and only if h is a polynomial (by the Borel-Carathéodory inequality or the lemma on the logarithmic derivative). The assertion on $T(r)$ follows from the subsequent theorem and $T(r, e^Q) \asymp r^{\deg Q}$. \square

Theorem. *If f and g share four values, at least one of them by counting multiplicities, then either the conclusion of the Four-Value-Theorem or else*

$$\frac{5}{209}T(r) \leq T(r, \Phi) + S(r) \leq 2T(r) - 2\overline{N}(r, \infty) + S(r)$$

holds.

Proof. We assume that $\mathbf{a}_4 = \infty$ is shared by counting multiplicities, and first suppose that $\Phi = \Phi_f/\Phi_g = \phi_f/\phi_g$ is non-constant. Then from

$$\begin{aligned} N(r, 1/\Phi_g) &\leq N(r, 1/\Psi) + S(r) = S(r), \\ N(r, 1/\phi_g) &= \overline{N}(r, \infty) + S(r) \\ m(r, \phi_g) &= S(r) \end{aligned}$$

(and the same for g replaced by f) follows the upper estimate

$$\begin{aligned} T(r, \Phi) = m(r, \Phi) + S(r) &\leq m(r, 1/\phi_g) + S(r) \\ &= N(r, \phi_g) - N(r, 1/\phi_g) + S(r) \\ &= \sum_{\nu=1}^3 \overline{N}(r, \mathbf{a}_\nu) - \overline{N}(r, \infty) + S(r) \\ &= 2T(r) - 2\overline{N}(r, \infty) + S(r). \end{aligned}$$

Now suppose that $f(z_0) = g(z_0) = \mathbf{a}_\nu$ holds with multiplicities ℓ_f and ℓ_g , say, hence $\Phi(z_0) = \ell_f/\ell_g$. Noting that the sequence of \mathbf{a}_ν -points with $\min\{\ell_f, \ell_g\} > 1$ has counting function $S(r)$, and restricting ℓ_f and ℓ_g to the range $\{1, \dots, \ell\}$, we obtain by Nevanlinna's second main theorem

$$\begin{aligned} T(r) &\leq \sum_{\nu=1}^3 \overline{N}(r, \mathbf{a}_\nu) + S(r) \\ &\leq (2\ell + 1)T(r, \Phi) + \frac{1}{\ell + 2} \sum_{\nu=1}^3 \left[N\left(r, \frac{1}{f - \mathbf{a}_\nu}\right) + N\left(r, \frac{1}{g - \mathbf{a}_\nu}\right) \right] + S(r) \\ &\leq (2\ell + 1)T(r, \Phi) + \frac{6}{\ell + 2}T(r) + S(r), \end{aligned}$$

hence

$$(8) \quad T(r) \leq \frac{(2\ell+1)(\ell+2)}{\ell-4} T(r, \Phi) + S(r) \quad (\ell > 4)$$

holds. The factor $\frac{209}{5}$ is obtained for $\ell = 9$. If, however, Φ is constant, then we have actually $\Phi \equiv \ell$ or $\Phi \equiv 1/\ell$ for some $\ell \in \mathbb{N}$. This yields, in the first case, say,

$$\prod_{\nu=1}^3 (f - \mathbf{a}_\nu) = C \prod_{\nu=1}^3 (g - \mathbf{a}_\nu)^\ell \quad (C \neq 0 \text{ some constant}).$$

From $T(r, f) \sim \ell T(r, g)$ then follows $\ell = 1$, hence f and g share all values \mathbf{a}_ν by counting multiplicities. \square

Combining the previous result with the second Key Observation, we obtain:

Corollary (Yi & Li [25]). *Suppose f and g share four values \mathbf{a}_ν , at least one of them by counting multiplicities. Then either both functions have infinite lower order or else have equal finite integer order that also equals the lower order.*

Remark. The authors of [25] proved in a way similar to ours the inequality

$$\frac{1}{77} T(r) \leq T(r, \Phi) + S(r) \leq 4T(r) + S(r).$$

Each of the counterexamples (by Gundersen, Reinders and the author) are (can be reduced to) either rational functions of e^z or else elliptic functions. Since elliptic functions have no deficient value, we obtain:

Corollary. *No pair of elliptic functions can share four values, at least one of them by counting multiplicities.*

Theorem. *Let f and g be $2\pi i$ -periodic functions of finite order sharing four values, one of them by counting multiplicities. Then f and g are rational functions of e^z .*

Proof. The functions $\Psi = \Phi_f \Phi_g$, Φ_f and Φ_g are entire of finite order and also $2\pi i$ -periodic. But Ψ is a polynomial, hence constant, and thus Φ_f and Φ_g are zero-free. This yields

$$(9) \quad \Phi_f(z) = e^{mz+c_f} \text{ and } \Phi_g(z) = e^{-mz+c_g}$$

for some $m \in \mathbb{Z} \setminus \{0\}$ and complex constants c_f and c_g . From $T(r, \Phi_f/\Phi_g) \sim \frac{2|m|}{\pi} r$, hence $T(r) \asymp r$ then follows that f and g are rational functions of e^z . \square

Remark. Generally spoken, meromorphic functions $h(z) = R(e^z)$, where R is rational with $\deg R = d > 1$, have Nevanlinna characteristic $T(r, h) \sim \frac{d}{\pi} r$ and deficient values $R(0)$ and $R(\infty)$, which may, of course, coincide. If, e.g., $R(u) \sim \mathbf{a} + bu^{-\rho}$ as $u \rightarrow \infty$, then the contribution of the right half plane to $m(r, 1/(f - \mathbf{a}))$ is $\sim \frac{\rho}{\pi} r$. In Gundersen's example we have $\delta(\mathbf{a}, f) = 1/2$ for $\mathbf{a} = 0, 1$, and $\delta(\mathbf{a}, g) = 1/2$ for $\mathbf{a} = \infty, -1/8$.

Let $f(z) = R(e^z)$ and $g(z) = S(e^z)$ share four values \mathbf{a}_ν ($\in \mathbb{C}$ for technical reasons), without assuming anything about multiplicities. Then Mues' function $\Psi = \Phi_f \Phi_g$

is a non-zero constant, and from $R(u) \sim \mathbf{a}_\mu + bu^{-\rho}$ and $S(u) \sim \mathbf{a}_\nu + cu^{-\sigma}$ ($bc \neq 0$, $\rho, \sigma > 0$) as $u \rightarrow \infty$ follows

$$\begin{aligned} \Phi_f(z) &\sim b_1(\mathbf{a}_\mu - \mathbf{a}_\nu) + O(e^{-\min\{\rho, \sigma\}\operatorname{Re} z}) \quad (b_1 \neq 0) \\ \Phi_f(z) &\sim c_1(\mathbf{a}_\mu - \mathbf{a}_\nu) + O(e^{-\min\{\rho, \sigma\}\operatorname{Re} z}) \quad (c_1 \neq 0) \end{aligned} \quad \text{as } \operatorname{Re} z \rightarrow +\infty.$$

Thus $\mathbf{a}_\mu \neq \mathbf{a}_\nu$, and a similar result near $u = 0$ shows that at least one of the values \mathbf{a}_ν (but no value $b \neq \mathbf{a}_\nu$) is deficient for f , and the same is true for g (and the same or some other value \mathbf{a}_μ). If \mathbf{a}_4 , say, is shared by counting multiplicities, then $\delta(\mathbf{a}_4, f) = \delta(\mathbf{a}_4, g) > 0$. More precisely we have $R(u) \sim \mathbf{a}_4 + bu^{-\rho}$ ($u \rightarrow \infty$), $S(u) \sim \mathbf{a}_4 + cu^\rho$ ($u \rightarrow 0$), $\delta(\mathbf{a}_4, f) = \delta(\mathbf{a}_4, g) = \rho/d$, and $R(0) = \mathbf{a}_\mu$, $S(\infty) = \mathbf{a}_\nu$ for some $\mu, \nu < 4$. We switch now to values $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4 = \infty$ and assume that \mathbf{a}_4 is shared by counting multiplicities, and also that $R(\infty) = S(0) = \infty$ holds. From $m(r, f) \sim m(r, g) \sim \frac{d}{\pi}r$ for some ρ ($1 \leq \rho < d$) we obtain

$$R(u) = \frac{P(u)}{Q(u)}, \quad S(u) = \frac{\tilde{P}(u)}{u^\rho Q(u)}, \quad \deg \tilde{P} \leq d \text{ and } \rho + \deg Q = \deg P = d;$$

the zeros of Q are simple and $\neq 0$.

7. Functions of finite order

Under certain circumstances it may happen that a problem for meromorphic function of arbitrary order of growth can be reduced to a problem for functions of finite order by the well-known Zalcman Lemma [26]. A prominent example can be found in [1]. This might also be the case here, although there are some obstacles, as will be seen later.

Rescaling Lemma. *Let f and g share the values \mathbf{a}_ν ($1 \leq \nu \leq 4$). Then either the spherical derivatives $f^\#$ and $g^\#$ are bounded on \mathbb{C} , or else there exist sequences $(z_k) \subset \mathbb{C}$ and $(\rho_k) \subset (0, \infty)$ with $\rho_k \rightarrow 0$, such that the sequences (f_k) and (g_k) , defined by*

$$f_k(z) = f(z_k + \rho_k z) \text{ and } g_k(z) = g(z_k + \rho_k z),$$

simultaneously tend to non-constant meromorphic functions \hat{f} and \hat{g} , respectively; \hat{f} and \hat{g} share the values \mathbf{a}_ν and have bounded spherical derivatives.

Proof. We assume that $f^\#$ is not bounded. Then the existence of the sequences (z_k) (tending to infinity) and (ρ_k) can be taken for granted for the function f by Zalcman's Lemma.

If we assume that the sequence (g_k) is not normal on \mathbb{C} , then again by Zalcman's Lemma there exist sequences $(k_\ell) \subset \mathbb{N}$, $\hat{z}_\ell \rightarrow z_0 \in \mathbb{C}$ and $\sigma_\ell \downarrow 0$, such that $\hat{g}_\ell = g_{k_\ell}(\hat{z}_\ell + \sigma_\ell z)$ tends to some non-constant meromorphic function \tilde{g} . Then on one hand, the corresponding sequence (\hat{f}_ℓ) tends to a constant, while on the other hand every limit function of (\hat{f}_ℓ) shares the values \mathbf{a}_ν with \tilde{g} by Hurwitz' Theorem. Since \tilde{f} assumes at least two of these values, our assumption on the sequence (g_k) was invalid. We may assume that (g_k) tends to \hat{g} . Then \hat{g} shares the values \mathbf{a}_ν with \hat{f} , hence is non-constant.

Finally, if we assume that $\hat{g}^\#$ is unbounded on \mathbb{C} , then we may apply the first argument to the functions \hat{g} and \hat{f} in this order, but now with the *a priori* knowledge that $\hat{f}^\#$ is bounded. Then some sequence $\hat{g}(\tilde{z}_k + \tau_k z)$ tends to some non-constant

limit, while a sub-sequence of the sequence $\hat{f}(\tilde{z}_k + \tau_k z)$ tends to a constant (the sequence of spherical derivatives is $O(\tau_k)$). This proves the Rescaling Lemma. \square

Remark. Till now, however, it cannot be excluded that

- (a) a shared value gets lost in the sense that it becomes a Picard value for \hat{f} and \hat{g} , although it is not for f and g ;
- (b) multiplicities get lost in the sense that \hat{f} and \hat{g} share some value \mathbf{a}_ν by counting multiplicities, although f and g do not;
- (c) $\hat{f} = \hat{g}$ (worst case).

In the third case everything is lost. But even if $\hat{f} \neq \hat{g}$ holds and the (3+1)-Conjecture turns out to be true for functions with bounded spherical derivative, we can only deduce $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = -1$, except if we are able to rule out also the first and second case. Nevertheless we proceed in this direction.

Theorem. *Suppose that f and g share the values \mathbf{a}_ν ($1 \leq \nu \leq 4$) and have bounded spherical derivative. Then Mues' function Ψ is a constant. Moreover, if one value is shared by counting multiplicities, then*

$$(10) \quad \Phi_f(z) = e^{Q(z)+c_f} \text{ and } \Phi_g(z) = e^{-Q(z)+c_g}$$

holds, with Q some polynomial of degree one or two.

Proof. From $f^\# + g^\# \leq C$ follows that f and g have finite order (actually the order is at most two), and by Nevanlinna's lemma on the logarithmic derivative, Ψ is a polynomial. We assume $\Psi(z) \sim cz^m$ as $z \rightarrow \infty$ for some $m \geq 1$ and $c \neq 0$, and consider, for some sequence $z_n \rightarrow \infty$, the functions

$$f_n(z) = f(z_n + z_n^{-m/2}z) \text{ and } g_n(z) = g(z_n + z_n^{-m/2}z),$$

which also share the values \mathbf{a}_ν and have Mues' function

$$(11) \quad \Psi_n(z) = z_n^{-m} \Psi(z_n + z_n^{-m/2}z) \sim c \quad (n \rightarrow \infty),$$

while obviously f_n and g_n tend to constants. If the sequence (z_n) , which is quite arbitrary, can be chosen in such a way that

$$f(z_n) \rightarrow b \neq \mathbf{a}_\nu \text{ and } g(z_n) \rightarrow b' \neq \mathbf{a}_\nu \quad (1 \leq \nu \leq 4),$$

then f_n and g_n tend to constants b and b' , hence Ψ_n tends to 0 in contrast to relation (11). Now for any sequence (\hat{z}_n) the sequences $\hat{f}_n(z) = f(\hat{z}_n + z)$ and $\hat{g}_n(z) = g(\hat{z}_n + z)$ are normal on \mathbb{C} , we may assume that (some sub-sequence of \hat{f}_n , again denoted by) \hat{f}_n tends to some *non-constant* limit function \hat{f} , e.g. by choosing $\hat{z}_n \rightarrow \infty$ such that $|\hat{f}'(\hat{z}_n)| = 1$. Since \hat{g}_n shares the values \mathbf{a}_ν with \hat{f}_n we may (by normality and Picard's theorem) also assume that $\hat{g}_n \rightarrow \hat{g} \neq \text{const}$. Now we choose z_0 such that $\hat{f}(z_0) = b \neq \mathbf{a}_\nu$ and $\hat{g}(z_0) = b' \neq \mathbf{a}_\nu$, and set $z_n = \hat{z}_n + z_0$.

Finally, since $\Psi = \Phi_f \Phi_g$ is constant, the entire functions Φ_f and Φ_g are zero-free and have order one or two (as do f and g), hence (10) holds with $\deg Q = 1$ or 2 . \square

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